THE NUMBER OF GENERIC SINGULARITIES*

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We announce bounds on the dimension and number of components of the Hilbert scheme $\mathrm{Hilb}^n P_r$ parametrizing 0-dimensional schemes of length n in r-space P_r, over an algebraically closed field k. We then discuss generic singularities. We thank M. Schlessinger for remarking that generic does not imply rigid, and our colleagues N. Greenleaf and R. Speiser for their comments.

§1. *The Hilbert Scheme*

In the following theorems $r > 2$ and char $k = 0$. We will denote $\mathrm{Hilb}^n P_r$ by H.

Theorem 1. *There are constants a, b depending only on r, such that*

$$an^{2-2/r} < \dim H < bn^{2-2/r}.$$

The lower bound is not new, and is given by a family of graded ideals [*2*]. The upper bound [*3*] was inspired by and still depends on Grauert's normal form for ideals in a power series ring (Satz 5 in [*1*]). The proof uses the combinatorial fact due essentially to Macaulay [*4*] that the number of generators needed in Grauert's normal form for an ideal of colength n in the power series ring $k[[x_1, \cdots, x_r]]$ is less than $r!n^{1-1/r}$.

We say a component C of H is *elementary* if it parametrizes only irreducible schemes. Otherwise C is *composite*. A subscheme Z of P_r parametrized by a generic point of C is the union of irreducible subschemes Z_i of lengths n_i with $\sum n_i = n$. Z_i is parametrized by a generic point of an elementary component C_i of $\mathrm{Hilb}^{n_i} P_r$. Thus

Lemma 1. *There is a one-to-one correspondence between components of H and the sets C_i such that each C_i is an elementary component of $\mathrm{Hilb}^{n_i} P_r$ and $\sum n_i = n$.*

Lemma 1 and the regularity of the growth of $\dim H$ allow us to show

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Theorem 2. *Let* $c = (b/a)^{r/r-2}$. *Given* n, *there is a number* N *between* n *and* $c^{-1}n$ *such that* $\mathrm{Hilb}^N P_r$ *has an elementary component.*

Proof. Suppose by way of contradiction there is no such elementary component. Then if C is a top-dimensional component of H corresponding to the set $\{C_i\}$ of elementary components, as in Lemma 1, each $n_i \leqq c^{-1}n$, and by Theorem 1,

$$\begin{aligned} an^{2-2/r} < \dim C \leqq \sum \dim C_i \leqq & \sum b(n_i)^{2-2/r} \\ & \leqq \left(\sum bn_i\right)(c^{-1}n)^{1-2/r} \\ & \leqq (bn)(c^{-1}n)^{1-2/r} \text{ since } \sum n_i = n \\ & \leqq bn^{2-2/r}c^{2/r-1}. \end{aligned}$$

Then $$a < bc^{2/r-1},$$

contradicting the choice of c. This shows Theorem 1 $\Rightarrow$ Theorem 2.

Corollary 1. *There are in all more than* $\log_c n$ *elementary components of*

$$\mathrm{Hilb}^1 P_r \cup \cdots \cup \mathrm{Hilb}^n P_r.$$

Theorem 2 and Lemma 1 show that the number of components of H is greater than $p(n)$, the number of partitions of n into parts $1, c, c^2, \cdots$. $p(n)$ satisfies $p(cn) = 1 + \sum_1^n p(t)$. We conclude

Theorem 3. *The number of components of* H *is greater than*

$$p(n) > \sum_{i=0}^{\infty} (n^i/i!)c^{i(i+1)/2}.$$

If $r = 3$, we may use $a = 1/40$, $b = 18$, and $c = (40 \cdot 18)^3$ in the theorems. We expect that Theorem 3 gives a poor bound. If, for example, there are several elementary components of some $\mathrm{Hilb}^N P_r$, we may replace $p(n)$ by $2^{n/N}$. Certainly, it should be possible to show that $\dim H = a'n^{2-2/r}(1 + O(n^{-s}))$, in which case the number of elementary components of H is at least $c'n^u$, with $u = s/(2 - 2/r)$.

§2. *Generic Singularities*

Suppose for simplicity C is an elementary component of H, that u is generic point of C, and that Z with support z in P_r is the scheme parametrized by u. If $n = 1$, Z is z and is non-singular; if $n > 1$, Z is a generic singularity at z, and is $\mathrm{Spec}(O_z/I)$ for an ideal I of colength n in the local ring O_z. There are two possibilities.

Case 1. Z is rigid.

Case 2. Z has a deformation Z'. Then Z is also a deformation of Z', no Z' is rigid, and every deformation Z' concentrated at z is $(\mathrm{Spec}(O_z/I')$ for an ideal I' having the *same type* as I. (See [1] or [3] for the definition of type.)

Schlessinger has given examples of non-rigid, generic singularities in higher dimensions [5]. No specific examples are known of generic 0-dimensional singularities, which hampers the effort to determine whether any are rigid!

Considering ideals of finite colength in $\mathbf{C}[[x,y]]$ (none of which is a generic singularity in the above sense), there are some types T such that a generic ideal I of that type has no deformations of the same type, like $T = (1,1,1,0,\cdots)$ and $I = (y + ax + bx^2, m^3)$, a, b transcendentals. For other types a generic ideal has deformations of the same type, like $T = (1,2,3,4,3,0,\cdots)$ and $I = (x^4 + ax^2y^2 + bxy^3 + cy^4, x^3y + dx^2y^2 + exy^3 + fy^4, m^5)$, with $a,\cdots,f$ transcendentals. This is verified simply by computing those tangents in $\mathrm{Hom}(I, A/I)$ arising from deforming $a,\cdots,f$ and determining whether each such tangent is $\alpha\frac{\partial\cdot}{\partial x} + \beta\frac{\partial\cdot}{\partial y}$ for some $\alpha,\beta = k[[x,y]][a,\cdots,f]$. If this behavior is imitated among the generic singularities, both rigid and non-rigid examples will occur.

REFERENCES

[1] GRAUERT, H., "Über die Deformation isolatier Singularitäten analystischer Mengen," *Inventiones Math.* **15** (1972), 171–198.

[2] IARROBINO, A., "Reducibility of the families of 0-dimensional schemes on a variety," *Inventiones Math.* **15** (1972), 72–77.

[3] ———, "Families of ideals in a ring of power series, I" (to appear).

[4] MACAULAY, F. S., "Some properties of enumeration in the theory of modular systems," *Proc. Lon. Math. Soc.*, Ser. 2, **26** (1927), 531–555.

[5] SCHLESSINGER, M., "Rigid singularities," pp. 147–162 in this volume.

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